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LONG MONOTONE PATHS ON SIMPLE 4-POLYTOPES

BY

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ABSTRACT

The *Monotone Upper Bound Problem* (Klee, 1965) asks if the maximal number $M(d,n)$ of vertices in a monotone path along edges of a d-dimensional polytope with n facets can be as large as conceivably possible: Is $M(d,n) = M_{\text{ubt}}(d,n)$, the maximal number of vertices that a d-polytope with n facets can have according to the Upper Bound Theorem?

We show that in dimension $d = 4$, the answer is "yes", despite the fact that it is "no" if we restrict ourselves to the dual-to-cyclic polytopes. For each $n \geq 5$, we exhibit a realization of a polar-to-neighborly 4-dimensional polytope with n facets and a Hamilton path through its vertices that is monotone with respect to a linear objective function.

This constrasts an earlier result, by which no polar-to-neighborly 6-dimensional polytope with 9 facets admits a monotone Hamilton path.

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1. Introduction

In 1965, Klee [4] posed the *Monotone Upper Bound Problem*: For $n > d \geq 2$, he asked for the maximal number $M(d,n)$ of vertices of a d-dimensional polytope with n facets that can lie on a *monotone path*, i.e., on a path along edges that is strictly increasing with respect to a linear objective function. His motivation came on the one hand from trying to understand the complexity of the simplex algorithm for linear programming, and on the other from McMullen's 1971 *Upper Bound Theorem* [5] (claimed by Motzkin [6] in 1957), which states that the maximal number $M_{\mathrm{ubt}}(d,n)$ of vertices that any d-dimensional polytope with n facets can have is achieved by the polars $C_d(n)^\Delta$ of cyclic d-polytopes with n facets.

The Upper Bound Theorem yields, for all $n > d \geq 2$, the inequality

$$M(d,n) \leq M_{\mathrm{ubt}}(d,n), \tag{1}$$

but from this it is not clear whether equality always holds, that is, if for all $n > d \geq 2$ one can construct a simple polar-to-neighborly d-polytope with n facets that admits a monotone Hamilton path with respect to a linear objective function. Equality in (1) *is* known in the cases $d \leq 3$ and $n \leq d+2$.

However, in [7] we show that in fact $M(6,9) < M_{\mathrm{ubt}}(6,9)$: there exists *no* realization of the (combinatorially unique) polar-to-neighborly 6-polytope $C_6(9)^\Delta$ with 9 facets and 30 vertices that admits such a monotone Hamilton path.

For $d = 4$ and $n = 8$, one can show using the same (basically combinatorial) methods that there is also no realization of $C_4(8)^\Delta$ with a monotone Hamilton path — but as we will show here, there are other dual-to-neighborly but not dual-to-cyclic 4-dimensional polytopes with 8 facets that admit a realization with a monotone path through all vertices.

In fact, in this paper we prove considerably more: we provide a geometric construction which shows that the inequality (1) is tight in dimension $d = 4$ for all $n \geq 5$.

Main Theorem: *For each integer $m \geq 0$, there exists a simple 4-dimensional polar-to-neighborly polytope Q_m with $n = m+5$ facets and a linear objective function $f\colon \mathbb{R}^4 \to \mathbb{R}$, such that the orientation induced by f on the 1-skeleton of Q_m admits a monotone Hamilton path. Therefore,*

$$M(4,n) = M_{\mathrm{ubt}}(4,n) = \frac{1}{2}n(n-3).$$

In other words, the maximal number $M(4,n)$ of vertices on a strictly monotone path in the graph of a 4-dimensional polytope with n facets equals the maximal *number of vertices that such a polytope can have according to the Upper Bound Theorem.*

An interesting feature used in our proof is that for $m \geq 3$, the polar-to-neighborly polytopes Q_m are not polar to cyclic ones. In fact, exhaustive enumeration shows that already the graph of $C_4(8)^\Delta$ does not satisfy a combinatorial condition necessary for the existence of an monotone path, namely, it does not admit a Hamilton AOF Holt–Klee orientation. This is also true for the graphs of the polytopes $C_4(n)^\Delta$ for $8 \leq n \leq 12$; we conjecture that the graphs of $C_4(n)^\Delta$ for all $n \geq 8$ admit no Hamilton AOF Holt–Klee orientation.

The structure of the paper is as follows: We first give an explicit description reminiscent of Gale's Evenness Criterion for polar-to-cyclic polytopes — of the combinatorial structure of a family $\{Q_m^d : d \geq 4 \text{ even}, m \geq 0\}$ of simple (polar-to-)neighborly d-dimensional polytopes with $m+d+1$ facets in Sections 2 and 3. For $d = 4$, we then use this description to specify a Hamilton path π_m on each $Q_m := Q_m^4$ (Section 4). In Section 5, we start with a monotone path π_0 on a certain realization of the 4-simplex Q_0, and for $m \geq 0$ inductively realize the polytope Q_{m+1} in such a way that the path π_{m+1} is strictly monotone with respect to a suitable objective function (Theorem 2.5). We proceed in three steps: First, we position Q_m in a suitable way with respect to the standard coordinates on $\mathbb{R}^4$ (Section 5.4). We then find a "cutting plane" H_{m+1} such that the polytope $Q_m \cap H_{m+1}^{\geq 0}$ has the right combinatorial type (Section 5.5). Finally, we complete the construction in Section 5.6 by applying a projective transformation ψ to $\mathbb{R}^4$ such that the path $\psi(\pi_m)$ on $Q_{m+1} := \psi(Q_m \cap H_{m+1}^{\geq 0})$ is strictly monotone with respect to the objective function $f\colon \mathbb{R}^4 \to \mathbb{R}$, $\mathbf{x} \mapsto x_4$.

2. Main results

THEOREM 2.1 (modified Gale's Evenness Criterion): *For each $m \geq 0$ and even $d \geq 4$, the following sets correspond to the vertices of a combinatorial type $\widetilde{Q}_m^d$ of a simple d-dimensional polar-to-neighborly polytope with $n = m + d + 1$ facets.*

▷ Type 1. *The union of one "triplet with a hole" and $d/2 - 1$ pairs of indices*

$$\{j_1, j_1 + 2\} \cup \{j_2, j_2 + 1\} \cup \cdots \cup \{j_{d/2}, j_{d/2} + 1\},$$

where $1 \leq j_1 < n - d + 1$, $j_1 + 3 \leq j_2$, $j_k + 2 \leq j_{k+1}$ for $2 \leq k \leq d/2 - 1$, and $j_{d/2} < n$.

- ▹ Type 2a. *The union of one triplet, the singleton* $\{n\}$, *and* $d/2-2$ *pairs of indices*

$$\{j_1, j_1+1, j_1+2\} \cup \{j_2, j_2+1\} \cup \cdots \cup \{j_{d/2-1}, j_{d/2-1}+1\} \cup \{n\},$$

where $1 \le j_1 < n-d+1$, $j_1+3 \le j_2$, $j_k+2 \le j_{k+1}$ *for* $2 \le k \le d/2-2$, *and* $j_{d/2-1} < n-1$.

- ▹ Type 2b. *The union of* $d/2$ *pairs of indices*

$$\{1,2\} \cup \{j_1, j_1+1\} \cup \cdots \cup \{j_{d/2-1}, j_{d/2-1}+1\},$$

where $3 \le j_1$, $j_k+2 \le j_{k+1}$ *for* $2 \le k \le d/2-2$, *and* $j_{d/2-1} < n$.

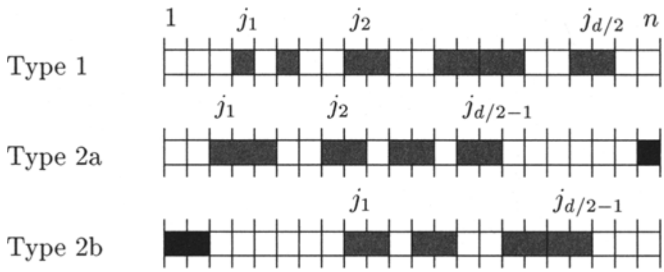

Figure 1. The vertex-facet incidences of the polytopes $\widetilde{Q}^d_m$ are obtained from these patterns by fixing the dark boxes, and sliding the lighter boxes between 1 and n without overlap. For Type 1, the box $\{j_1, j_1+2\}$ must be regarded as one rigid unit.

Remark 2.2: If we accept for the moment the existence of the polytopes $\widetilde{Q}^d_m$, it is easy to verify that they are polar-to-neighborly by counting the number of vertices using Figure 1:

$$\begin{aligned} f_0(\widetilde{Q}^d_{n-d-1}) &= \underbrace{\binom{n-2-(d/2-1)}{d/2}}_{\text{Type 1}} + \underbrace{\binom{n-2-(d/2-2)-1}{d/2-1}}_{\text{Type 2a}} \\ &\quad + \underbrace{\binom{n-2-(d/2-1)}{d/2-1}}_{\text{Type 2b}} \\ &= \binom{n-1-d/2}{d/2} + 2\binom{n-1-d/2}{d/2-1} \\ &= \binom{n-d/2}{d/2} + \binom{n-1-d/2}{d/2-1}, \end{aligned}$$

which is the number of vertices of a simple polar-to-neighborly d-polytope with $n = m + d + 1$ facets, since d is assumed even. By [9, Chapter 8], *any* polytope with that many vertices is polar-to-neighborly. ∎

From now on, we will always write $\widetilde{Q}_m := \widetilde{Q}^4_m$.

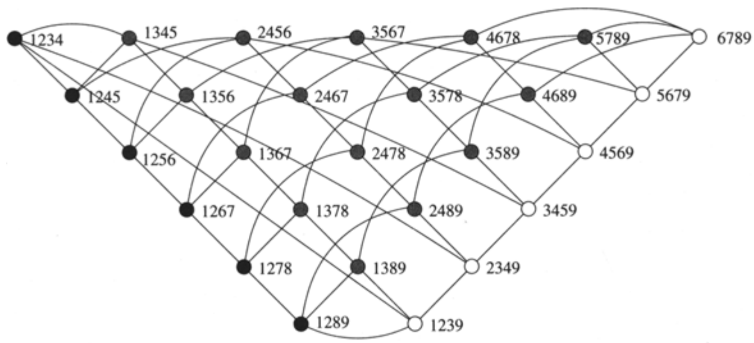

Figure 2. Graph of the 4-polytope $\widetilde{Q}_4$ with $n = 9$ facets. Vertices of type 1, 2a, and 2b are drawn in gray, white, and black, respectively. Each vertex is labelled with the facets it is incident to.

Proposition 2.3: *Each polytope $\widetilde{Q}_m$ admits a Hamilton path $\widetilde{\pi}_m$ in its graph that induces an AOF-orientation* (cf. Figure 3 and Definition 4.1 below).

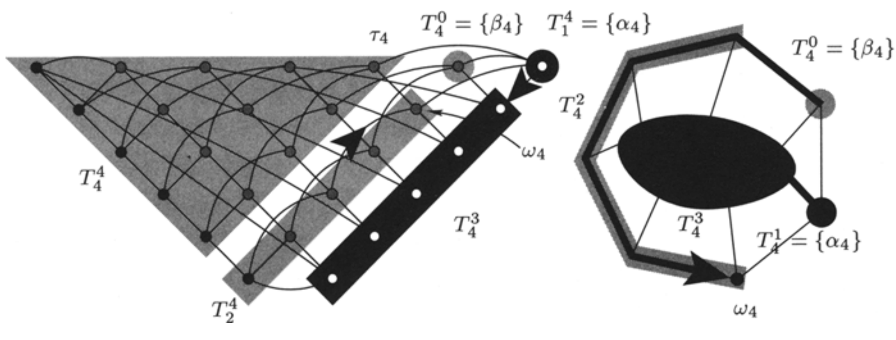

Figure 3. *Left:* Graph of $\widetilde{Q}^4_4$. The partition of the vertices into the tips T^0, $T^1, \dots, T^4$ is shown, along with the Hamilton path $\widetilde{\pi}_4$ (bold). The source α_m is labeled $\{n-3, n-2, n-1, n\}$, and the sink $\omega_m = \{n-5, n-3, n-1, n\}$ (here $m = 4$ and $n = 9$). See Convention 5.1 for the labels of the other marked vertices. *Right:* The facet F^3_4 with the restriction of $\widetilde{\pi}_4$ to it.

Remark 2.4: The crucial property for our realization construction is that the path $\widetilde{\pi}_m$ begins in a certain facet F_m^3 of the polytope Q_m (defined below), traverses the rest of Q_m, and then returns to F_m^3 (cf. Figure 3). This permits us to add new vertices to the beginning and end of $\widetilde{\pi}_m$ by modifying only the facet F_m^3.

THEOREM 2.5: *There exists a family $\{Q_m : m \geq 0\}$ of special realizations of the combinatorial types $\widetilde{Q}_m$, in which each Hamilton path π_m visits the vertices of Q_m in the order given by increasing x_4-coordinate. This family may be realized inductively starting from the 4-simplex Q_0 in such a way that for all $m \geq 0$, a realization of Q_{m+1} with a monotone Hamilton path π_{m+1} may be obtained from any realization of Q_m with such a path π_m.*

3. Constructing the combinatorial types $\widetilde{Q}_m^d$

3.1. FACET SPLITTING. We will prove Theorem 2.1 using Barnette's technique of *facet splitting* [1]. Put briefly, for each even $d \geq 4$ we will inductively construct a family $\{(\widetilde{Q}_m^d, \mathcal{F}_m) : m \geq 0\}$, where each $\widetilde{Q}_m^d$ is the combinatorial type of a simple d-dimensional polytope with $m + d + 1$ facets, and $\mathcal{F}_m$ is a flag of faces on $\widetilde{Q}_m^d$ (to be defined shortly). We then use $\mathcal{F}_m$ to find a "good" oriented hyperplane H_{m+1} in general position with respect to the vertices of $\widetilde{Q}_m^d$, and set $\widetilde{Q}_{m+1}^d := \widetilde{Q}_m^d \cap H_{m+1}^{\geq 0}$.

Definition 3.1: Let P be a d-dimensional simple polytope. A **flag of faces** on P is a chain

$$\mathcal{F}\colon \emptyset = F^{-1} \subset F^0 \subset F^1 \subset \cdots \subset F^d = P \tag{2}$$

of faces of P such that $\dim F^i = i$ for $i = 0, 1, \ldots, d$. The **i-th tip** of a flag $\mathcal{F}$ is $T^i := \operatorname{vert} F^i \setminus \operatorname{vert} F^{i-1}$, for $0 \leq i \leq d$. We say that the tip T^i is **even** resp. **odd** according to the parity of i. Moreover, for $0 \leq k \leq d$ we set

$$T_{\text{even}}^{\leq k} = \bigcup_{\substack{0 \leq e \leq k \\ e \text{ even}}} T^e \quad \text{and} \quad T_{\text{odd}}^{\leq k} = \bigcup_{\substack{1 \leq o \leq k \\ o \text{ odd}}} T^o.$$

LEMMA 3.2: *Let P be a simple d-dimensional polytope with n facets, and $\mathcal{F}$ a flag of faces as in (2). Then there exists an affine oriented hyperplane H in general position with respect to P such that $T_{\text{even}}^{\leq d} \subset H^+$ and $T_{\text{odd}}^{\leq d} \subset H^-$. In particular, $P \cap H^{\geq 0}$ is a simple d-polytope with $n + 1$ facets.*

Proof: Pick an oriented point $\{v\} = H^0 \subset \operatorname{relint} F^1$ such that $T^0 \in (H^0)^+$. Inductively, for $1 \leq k \leq d-1$, if we have already chosen an oriented $(k-1)$-dimensional affine subspace H^{k-1} in aff F^k such that

$$T^{\leq k}_{\text{even}} \subset (H^{k-1})^+ \quad \text{and} \quad T^{\leq k}_{\text{odd}} \subset (H^{k-1})^-, \tag{3}$$

we take a k-plane H^k that initially coincides with aff F^k, and orient it in such a way that T^{k+1} lies in $(H^k)^+$ if $k+1$ is even, respectively in $(H^k)^-$ if $k+1$ is odd. Now we rotate H^k by a sufficiently small amount around H^{k-1} in such a way that $T^{\leq k}_{\text{even}} \subset (H^k)^+$. Then (3) even holds with k replaced by $k+1$. By construction, the hyperplane $H := H^{d-1}$ is in general position with respect to P. ∎

Definition 3.3: The family $\{(\widetilde{Q}^d_m, \mathcal{F}_m) : m \geq 0\}$ of d-dimensional polytopes $\widetilde{Q}^d_m$ equipped with flags $\mathcal{F}_m$ of faces is defined in the following way:

(a) $\widetilde{Q}^d_0$ is the combinatorial type of the d-simplex $\operatorname{conv}\{v_1, v_2, \ldots, v_{d+1}\}$. The flag $\mathcal{F}_0$ is defined by setting $F^i_0 := \operatorname{conv}\{v_1, v_2, \ldots, v_{i+1}\}$ for $i = 0, 1, \ldots, d$. Then

$$T^i_0 := \operatorname{vert}(F^i_0) \setminus \operatorname{vert}(F^{i-1}_0) = \{v_{i+1}\} \quad \text{for } i = 0, 1, \ldots, d. \tag{4}$$

(b) For $m \geq 0$, let $H = H_{m+1}$ be the oriented hyperplane given by applying Lemma 3.2 to $P = \widetilde{Q}^d_m$ and $\mathcal{F} = \mathcal{F}_m$, and set $\widetilde{Q}^d_{m+1} := \widetilde{Q}^d_m \cap H^{\geq 0}_{m+1}$ and (cf. Figure 4)

$$\begin{aligned}
T^0_{m+1} &:= \operatorname{vert}(\operatorname{conv}(T^1_m \cup T^2_m) \cap H_{m+1}),\\
T^1_{m+1} &:= \operatorname{vert}(\operatorname{conv}(T^0_m \cup T^1_m) \cap H_{m+1}),\\
T^j_{m+1} &:= \operatorname{vert}\left(\operatorname{conv}\left(T^{j+1}_m \cup \bigcup_{\substack{0 \leq k < j \\ k+j \equiv 0 \bmod 2}} T^k_m\right) \cap H_{m+1}\right) \quad \text{for } j = 2, 3, \ldots, d-1,\\
T^d_{m+1} &:= \bigcup_{0 \leq k \leq d/2} T^{2k}_m.
\end{aligned}$$

The flag $\mathcal{F}_{m+1}$ is now defined by

$$F^j_{m+1} := \bigcup_{i=0}^{j} T^i_{m+1} \quad \text{for } j = 0, 1, \ldots, d.$$

Moreover, put

$$T^{\leq k}_{\text{even}}(m) = \bigcup_{\substack{0 \leq e \leq k \\ e \text{ even}}} T^e_m \quad \text{and} \quad T^{\leq k}_{\text{odd}}(m) = \bigcup_{\substack{1 \leq o \leq k \\ o \text{ odd}}} T^o_m.$$

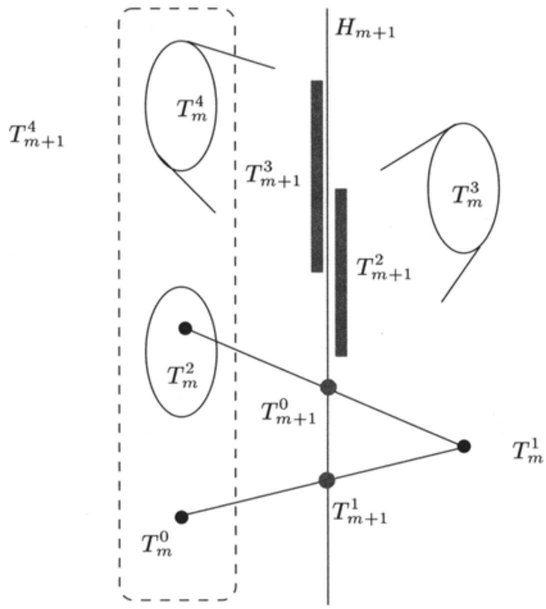

Figure 4. New tips in the case $d = 4$.

Remark 3.4:

(a) The polytopes $C_d(n)^\Delta$ arise by exchanging the definitions of T^0_{m+1} and T^1_{m+1}.

(b) All new vertices arise as the intersection of H_{m+1} with some edge $\operatorname{conv}\{v, w\}$ of $\widetilde{Q}^d_m$, where v and w lie in tips of different parity. Furthermore, all vertices of $\widetilde{Q}^d_m$ belonging to even tips are also vertices of $\widetilde{Q}^d_{m+1}$, and vertices in odd tips disappear.

PROPOSITION 3.5: *For each $m \geq 0$, the following is true for the pair $(\widetilde{Q}^d_m, \mathcal{F}_m)$:*

(a) *For all $i, j \in \mathbb{N}$ with $0 \leq i < j \leq d$ and $i + j = 1 \bmod 2$ and all $v \in T^i_m$, there is exactly one $w \in T^j_m$ such that $\operatorname{conv}\{v, w\} \in \operatorname{sk}^1(\widetilde{Q}^d_m)$. This gives rise to bijections $T^{\leq k}_{\text{even}}(m) \cong T^k_{m+1}$ for odd $0 < k < d$ resp. $T^{\leq k}_{\text{odd}}(m) \cong T^k_{m+1}$ for even $0 \leq k \leq d$.*

(b) *$|T^e_m| = |T^{e+1}_m| = \binom{e/2+m}{m}$ for even $e = 0, 2, \ldots, d-2$, and $|T^d_m| = \binom{d/2+m}{m}$. This proves again that $\widetilde{Q}^d_m$ is polar-to-neighborly.*

Proof: (a) This follows because v lies in $F^{j-1}_m = \bigcup_{i=0}^{j-1} T^i_m$, and $\operatorname{conv}(F^{j-1}_m)$ is a $(j-1)$-dimensional face of the simple polytope $\operatorname{conv}(F^j_m) = \operatorname{conv}(F^{j-1}_m \cup T^j_m)$.

(b) We proceed by induction, and can assume that the assertion holds for $m \geq 0$. From the bijections in part (a), we conclude for all even $e = 0, 2, \ldots, d-2$ that

$$|T^{e+1}_{m+1}| = |T^{e}_{m+1}| = \sum_{\substack{i=0 \\ i \text{ even}}}^{e} |T^{i}_{m}| = \sum_{k=0}^{e/2} |T^{2k}_{m}| = \sum_{k=0}^{e/2} \binom{k+m}{m} = \binom{e/2+m+1}{m+1}.$$

The calculation for $|T^{d}_{m+1}|$ is similar. The fact that $\widetilde{Q}^d_m$ is polar-to-neighborly follows by the same argument as in Remark 2.2, since

$$\begin{aligned} f_0(\widetilde{Q}^d_m) &= \sum_{k=0}^{d/2} \binom{k+m}{m} + \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} \binom{k+m}{m} \\ &= \binom{m+d/2+1}{d/2} + \binom{m+\lfloor (d-1)/2 \rfloor+1}{\lfloor (d-1)/2 \rfloor} \\ &= \binom{n-d/2}{d/2} + \binom{n-\lceil (d-1)/2 \rceil-1}{\lfloor (d-1)/2 \rfloor}. \end{aligned}$$ ∎

3.2. Combinatorics of the family $\widetilde{Q}^d_m$.

Convention 3.6: We introduce labelings to make the combinatorics of the $\widetilde{Q}^d_m$ explicit:

(a) For any labeling of the facets of a simple d-polytope P with labels in $[n] := \{1, 2, \ldots, n\}$, let $\lambda : \operatorname{vert} P \to \binom{[n]}{d}$ assign to each vertex v of P the set of labels of all facets that v is incident to. We identify a vertex v with its label $\lambda(v)$.

(b) The facets of the d-simplex $\widetilde{Q}^d_0$ on the vertex set $\{v_1, v_2, \ldots, v_{d+1}\}$ are labeled in such a way that $v_1 \equiv \lambda(v_1) = [d+1] \setminus \{2\}$, $v_2 = [d+1] \setminus \{1\}$, and $v_j = [d+1] \setminus \{j\}$ for $j = 3, 4, \ldots, d+1$ (cf. Figure 5).

(c) The "new" facet $\widetilde{Q}^d_m \cap H_{m+1}$ of $\widetilde{Q}^d_{m+1}$ is labelled $m+d+2$.

$\{v_5\} = T^4_0$ 2b 12|34

123|5 2a $T^3_0 = \{v_4\}$

$\{v_3\} = T^2_0$ 2b 12|45

234|5 2a $T^1_0 = \{v_2\}$

$\{v_1\} = T^0_0$ 1 13|45

Figure 5. The labeling of the vertices of the 4-simplex $\widetilde{Q}_0$ according to part (b) of Convention 3.6. Also shown is the classification of the vertices into types 1, 2a, 2b as in Proposition 3.7.

PROPOSITION 3.7: *Let $m \geq 0$ and $n = m + d + 1$. A vertex v of $\widetilde{Q}^d_m$ lies in T^i_m exactly if*

$$\max_n \overline{v} := \max([n] \setminus v) = \begin{cases} m+2 & \text{for } i = 0, \\ m+1 & \text{for } i = 1, \\ m+i+1 & \text{for } 2 \leq i \leq d. \end{cases}$$

Proof: This is true for $m = 0$ by (4) and Convention 3.6; see also Figures 5 and 6. For $m > 0$ and $2 \leq i \leq d-1$, the statement follows because any vertex $\tilde{v} \in T^i_m$ is of the form $\tilde{v} = \operatorname{conv}\{v, w\} \cap H_m \equiv (v \cap w) \cup \{n\}$ for some $v \in T^k_{m-1}$ and $w \in T^{i+1}_{m-1}$ with $k \leq i$. But then by induction,

$$\max_{n-1} \overline{v} < \max_{n-1} \overline{w} = (m-1) + (i+1) + 1 = m + i + 1,$$

so $\max([n] \setminus \tilde{v}) = m + i + 1$ as required. The case $i = d$ follows directly from Definition 3.3, and the cases $i = 0, 1$ are checked similarly. ∎

Proof of Theorem 2.1: The existence of the family $\{(\widetilde{Q}^d_m, \mathcal{F}_m) : m \geq 0\}$ follows from Lemma 3.2. Using Propositions 3.5 and 3.7, it is somewhat tedious but elementary to verify that for all $m \geq 0$, the vertices of $\widetilde{Q}^d_m$ are of the given types. More precisely, all vertices of $T^{\leq d}_{\text{even}}(m)$ are of type 1 or 2b, and $T^{\leq d-1}_{\text{odd}}(m)$ is made up entirely of vertices of type 2a; cf. Figure 6. ∎

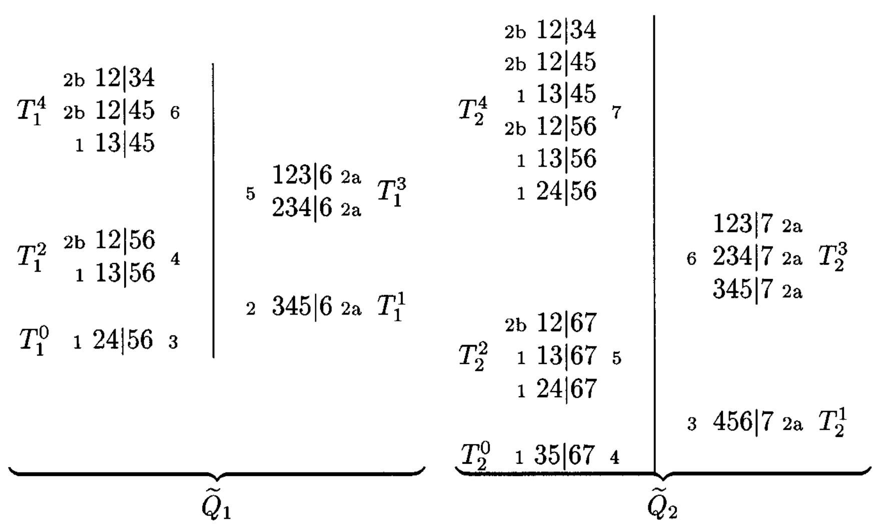

Figure 6. Vertex labels in the polytopes $\widetilde{Q}_1$ (left) and $\widetilde{Q}_2$ (right). Also shown are the type (outside) of each vertex v and the value of $\max_n \bar{v}$ (inside).

4. A Hamilton path $\widetilde{\pi}_m$ that induces an AOF-orientation on $\widetilde{Q}_m$

Definition 4.1: Let P be a simple d-polytope. An acyclic orientation of the graph of P that has a unique sink in each face (including P itself) is called an **AOF-orientation** on P. For any orientation $\mathcal{O}$ of the graph of P and $0 \leq k \leq d$, denote by $h_k(\mathcal{O})$ the number of vertices of in-degree k in $\mathcal{O}$.

PROPOSITION 4.2 (see e.g. [9, Chap. 8.3] and [3]): *An acyclic orientation $\mathcal{O}$ of the graph of a simple d-polytope P is an AOF-orientation if and only if the h-vector of P coincides with the vector $(h_0(\mathcal{O}), h_1(\mathcal{O}), \ldots, h_d(\mathcal{O}))$.*

Proof of Proposition 2.3: By inspection of Figures 2 and 3, the algorithm of Figure 7 yields a Hamilton path $\widetilde{\pi}_m$ in the graph of $\widetilde{Q}_m$. Note that $\widetilde{\pi}_m$ passes through T^1_m, T^3_m, T^4_m, T^0_m, and T^2_m, in this order (cf. Remark 2.4).

(1) *"Odd stage".* **for** i **from** $n-3$ **to** 1 **do**
    visit $\{i, i+1, i+2, n\}$;
(2) *"Even stage".* **for** j **from** 3 **to** $n-1$ **do**
    $i := j-3$;
    while $i \geq 1$ **do** *"down" phase*
        visit $\{i, i+2, j, j+1\}$;
        $i := i-2$;
    visit $\{1, 2, j, j+1\}$;
    if j is even **then** $i := 2$; **else** $i := 1$;
    while $i \leq j-4$ **do** *"up" phase*
        visit $\{i, i+2, j, j+1\}$;
        $i := i+2$;

Figure 7. A Hamilton path $\widetilde{\pi}_m$ on the graph of $\widetilde{Q}_m$ that induces an AOF-orientation ($n := m+5$).

We now verify that $\widetilde{\pi}_m$ induces an AOF-orientation on the graph of $\widetilde{Q}_m$. The h-vector of a simple polar-to-neighborly d-dimensional polytope with $n = m+d+1$ facets is given by $h_k = \binom{n-d-1+k}{k} = \binom{m+k}{k}$ for $k = 0, 1, \ldots, d$. Therefore, by Proposition 3.5,

$$(|T^1_m|, |T^3_m|, |T^4_m|, |T^2_m|, |T^0_m|) = (h_0, h_1, h_2, h_3, h_4).$$

By Proposition 4.2, it suffices to verify using Figure 3 that if the orientation of each edge of the graph of $\widetilde{Q}_m$ is consistent with the total ordering induced by $\widetilde{\pi}_m$, then the vertices of T^1, T^3, resp. T^4 all have in-degree 0, 1 resp. 2, furthermore T^0 and all but one of the vertices of T^2 have in-degree 3, and this vertex, the sink, has in-degree 4. ∎

5. Realizing the monotone Hamilton paths

In this section we prove Theorem 2.5, and therefore our Main Theorem.

5.1. OUTLINE OF THE INDUCTIVE CONSTRUCTION. For all $m \geq 0$, we first find an oriented hyperplane H_{m+1} that separates the **odd part** $T^{\leq 4}_{\text{odd}}(m) = T^1_m \cup T^3_m$ from the **even part** $T^{\leq 4}_{\text{even}}(m) = T^0_m \cup T^2_m \cup T^4_m$ of π_m. We then create an intermediate pair $(Q'_{m+1}, \mathcal{F}'_{m+1})$ as in Proposition 3.5: $Q'_{m+1} := Q_m \cap H^{\geq 0}_{m+1}$ is a simple polar-to-neighborly polytope of the same combinatorial type as $\widetilde{Q}_{m+1}$, and the flag $\mathcal{F}'_{m+1}$ of faces is defined as in Definition 3.3(b).

Our combinatorial model $\widetilde{Q}_{m+1}$ provides us with a Hamilton path π_{m+1} on Q'_{m+1} that is not yet monotone with respect to the objective function $f : \mathbf{x} \mapsto x_4$. However, we will choose H_{m+1} in such a way that there exists a pencil

$$\mathcal{H} = \{H_t : t \in \mathbb{P}^1(\mathbb{R}) \cong \mathbb{R} \cup \{\infty\}\}$$

of hyperplanes in $\mathbb{R}^4$ with the following properties:

(S1) The intersection of all hyperplanes in $\mathcal{H}$ is a 2-flat $R = \bigcap_{t\in\mathbb{P}^1(\mathbb{R})} H_t$ (the **axis** of $\mathcal{H}$), and $\operatorname{vert} Q'_{m+1} \cap R = \emptyset$.

(S2) The pencil $\mathcal{H}$ "sorts the vertices of Q'_{m+1} correctly": If $p \in H_r$ and $q \in H_s$ are vertices of Q'_{m+1} with $r, s \neq \infty$ and p precedes q in π_{m+1}, then $r < s$.

We then apply a projective transformation ψ to $\mathbb{R}^4 \subset \mathbb{P}^4(\mathbb{R})$ that maps H_∞ to the hyperplane at infinity. Because the common intersection R of all hyperplanes in $\mathcal{H}$ is also mapped to infinity, the image $\psi(\mathcal{H}^b) = \psi(\mathcal{H}\backslash H_\infty) = \{\psi(H_t) : t \in \mathbb{R}\}$ is a family of parallel affine hyperplanes in $\mathbb{R}^4$. The new objective function f is then defined by the common normal vector to the hyperplanes in $\psi(\mathcal{H}^b)$, and the Hamilton path $\psi(\pi_{m+1})$ on $Q_{m+1} := \psi(Q'_{m+1})$ is strictly monotone with respect to f_{m+1} by (S2).

5.2. PROPERTIES OF THE FAMILY OF POLYTOPES.

Notation 5.1: We use the following names for some special vertices of $\widetilde{Q}_m$:

- ▷ The source $\{n-3, n-2, n-1, n\}$ of $\widetilde{\pi}_m$ is called α_m (so that $T^1_m = \{\alpha_m\}$).
- ▷ The sink is $\omega_m := \{n-5, n-3, n-1, n\} \in T^2_m$.
- ▷ $\beta_m := \{n-4, n-2, n-1, n\}$ (so that $T^0_m = \{\beta_m\}$).
- ▷ $\tau_m := \{n-5, n-3, n-2, n-1\} \in T^4_m$.

PROPOSITION 5.2:

(a) *The induced subgraph of* $sk^1(Q_m)$ *on* $T_m^1 \cup T_m^3$ *is a path of length* $m+1$ *on the* $m+2$ *vertices* $v_0^m = \alpha_m, v_1^m, \ldots, v_{m+1}^m$*, and the induced subgraph on* T_m^2 *is a path* $w_1^m, w_2^m, \ldots, w_{m+1}^m$.

(b) *For* $0 \le i \le m$*, the edge* $e_i = \mathrm{conv}\{v_i^m, v_{i+1}^m\}$ *in* T_m^3 *is incident to a 2-face* G_i *of* Q_m *such that the vertices of* $G_i \setminus e_i$ *are consecutive in* $\pi_m \cap T_m^4$.

(c) *For* $1 \le i \le m$*, the edge* f_i *of* Q_m *that connects* w_i^m *and* w_{i+1}^m *in* $T_m^2 \cap \pi_m$ *is incident to a quadrilateral* R_i *whose other two vertices are consecutive in* $T_m^4 \cap \pi_m$.

(d) *Set* $G(m) = \mathrm{vert} \bigcup_{i=0}^m G_i \setminus e_i$ *and* $R(m) = \mathrm{vert} \bigcup_{i=1}^m R_i \setminus f_i$*. Then* $G(m) \cup R(m) = T_m^4$*, and* $G(m) \cap R(m) = \tau_m$.

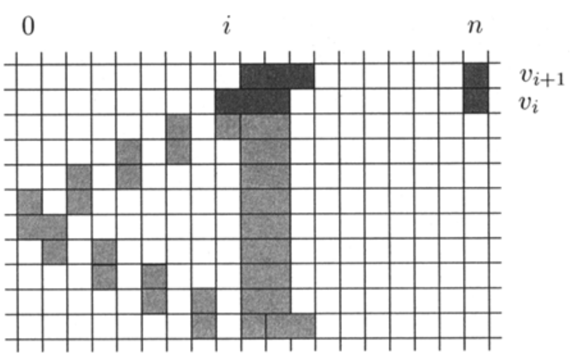

Figure 8. Vertices of a 2-dimensional face incident to $v_i = \{i, i+1, i+2, n\}$ and $v_{i+1} = \{i+1, i+2, i+3, n\}$ (dark) in $T_m^1 \cup T_m^3$. The light vertices lie in T_m^4 and form a subpath of π_m.

Proof: (a) All vertices of T_m^3 are of the form $\{i, i+1, i+2, n\}$ for $1 \le i \le n-3$, and the only way for two such vertices v_i^m and v_j^m to be adjacent for $i < j$ is to have $j = i+1$. The statement about the w_i^m follows in a similar way. (b) For $1 \le i \le m+1$, the 2-face incident to $v_{m+2-i} = \{i, i+1, i+2, n\}$ and $v_{m+1-i} = \{i+1, i+2, i+3, n\}$ that is the intersection of the facets $i+1$ and $i+2$ consists of the vertices of Figure 8. The claim (b) follows because these vertices form a contiguous segment of π_m, and (c) and (d) from Figure 9 (left). ∎

Observation 5.3: The new start vertex α_{m+1} of $\widetilde{\pi}_{m+1}$ lies on $\mathrm{conv}\{\alpha_m, \beta_m\}$, the new end vertex ω_{m+1} on $\mathrm{conv}\{v_1^m, \beta_m\}$, and β_{m+1} on $\mathrm{conv}\{\alpha_m, \omega_m\}$; see Figure 9 (right).

5.3. Start of the induction and inductive invariant. We work in $\mathbb{R}^4$ with standard coordinate vectors e_1, e_2, e_3, e_4, and abbreviate a linear subspace $_{\mathbb{R}}\langle e_i : i \in I\rangle$ by $\langle i : i \in I\rangle$ for all $I \subset \{1,2,3,4\}$. An essential tool will be **shear transformations**: these are linear maps $\sigma^a_{i,j} : \mathbb{R}^4 \to \mathbb{R}^4$ for $i, j \in \{1,2,3,4\}$, $i \neq j$, and $a \in \mathbb{R}$ whose matrix is $I_4 + a\delta_{i,j}$ with respect to the standard basis of $\mathbb{R}^4$. Here I_4 is the 4×4 unit matrix and $\delta_{i,j}$ is the 4×4 matrix whose only nonzero entry is a 1 in position (i, j). In particular, $\sigma^a_{i,j}$ maps e_i to $e_i + ae_j$, and the standard basis vectors e_k, $k \neq i$, to themselves.

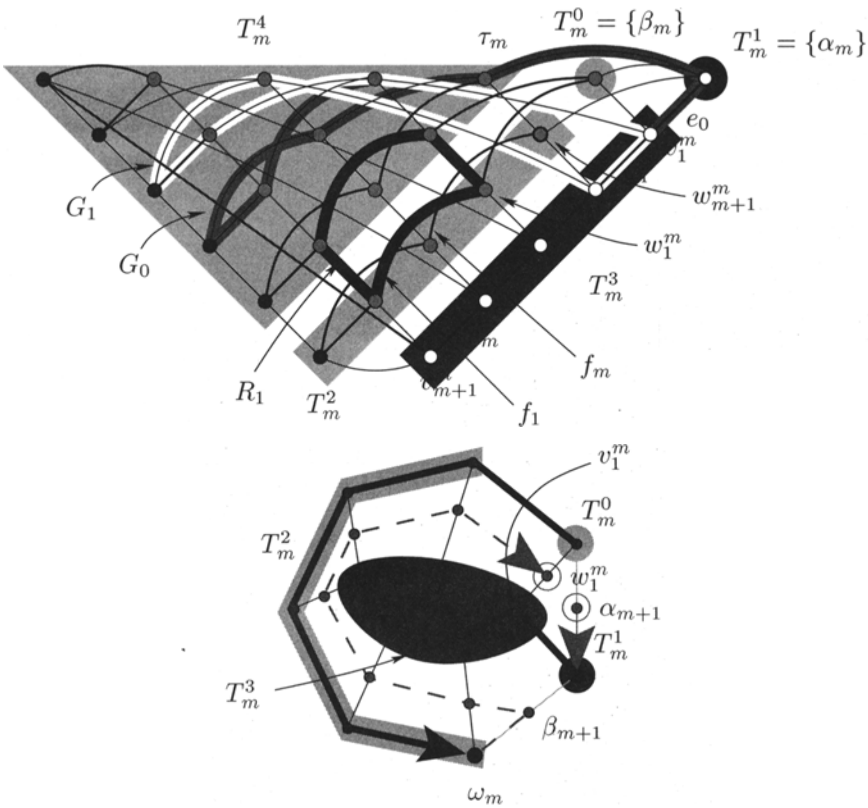

Figure 9. *Top:* More details about the graph of Q_m. We have highlighted the graphs of the 2-faces G_0 and G_1 that correspond to the edges e_0 and e_1 by Proposition 5.2 (b), and the 2-face R_1 that corresponds to the edge f_1 according to Proposition 5.2 (c). *Bottom:* The portion of the new Hamilton path $\widetilde{\pi}_{m+1}$ in the facet F^3_m.

The start of the induction is the pair $(Q_0, \mathcal{F}_0)$, where Q_0 is the 4-simplex whose vertices v_1, v_2, v_3, v_4, v_5 are given by the columns of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -3 & -1 & 3 & 2 & 1 \\ -2 & -1/2 & 0 & 1/4 & 2 \end{pmatrix}, \tag{5}$$

and $\mathcal{F}$ is the flag $\mathcal{F}_0$: $F_0^0 \subset F_0^1 \subset \cdots \subset F_0^4 = Q_0^4$ of faces labeled as in Definition 3.3. In particular, the vertices v_i lie in the following tips,

v_1	v_2	v_3	v_4	v_5
T_0^1	T_0^3	T_0^4	T_0^0	T_0^2

$F_0^2 = \operatorname{conv}\{v_1, v_4, v_5\}$, $F_0^3 = \operatorname{conv}\{v_1, v_2, v_4, v_5\}$, and $\pi_0 = (v_1, v_2, v_3, v_4, v_5)$.

For all $m \geq 0$ the polytopes Q_m will maintain the following property:

(M1) The Hamilton path π_m in the 1-skeleton of Q_m is strictly monotone with respect to the objective function f: $\mathbb{R}^4 \to \mathbb{R}$, $\mathbf{x} \mapsto x_4$.

5.4. Induction step I: Positioning the polytope. In this and the following section, we will position the polytope Q_m in such a way that the coordinate subspaces of $\mathbb{R}^4$ are compatible with the flag $\mathcal{F}_m$. More precisely,

- ▷ $F_m^3 = Q_m \cap \{\mathbf{x} \in \mathbb{R}^4 : x_1 = 0\}$, and $T_m^4 \subset \{\mathbf{x} \in \mathbb{R}^4 : x_1 > 0\}$; and
- ▷ the hyperplane $H_S = \{\mathbf{x} \in \mathbb{R}^4 : x_3 = 0\}$ will separate $T_{\text{even}}^{\leq 4}(m)$ from $T_{\text{odd}}^{\leq 4}(m)$.

Lemma 5.4: *Let π be the linear projection $\pi : \mathbb{R}^4 \to \langle 3, 4\rangle$, and use the notation of Convention 5.1 and Proposition 5.2(a). Then there exists a non-singular affine transformation σ of $\mathbb{R}^4$ such that $Q_m \equiv \sigma(Q_m)$ satisfies the following additional conditions, while $\pi_m \equiv \sigma(\pi_m)$ still satisfies (M1):*

(M2) $F_m^2 \subset \{\mathbf{x} \in \mathbb{R}^4 : x_1 = 0\}$.

(M3) $\operatorname{aff} F_m^3 = \{\mathbf{x} \in \mathbb{R}^4 : x_1 = 0\}$ *and* $Q_m \subset \{\mathbf{x} \in \mathbb{R}^4 : x_1 \geq 0\}$.

(M4) $(\alpha_m)_2 = 0$, $r_2 < 0$ *for all* $r \in F_m^2 \setminus \{\alpha_m\}$, *and* $(\beta_m)_2 < (v_1^m)_2$.

(M5) *The image of* F_m^2 *under* π *is full-dimensional:* $\dim \operatorname{aff}(\pi(F_m^2)) = 2$.

(M6) *The 3-flat* $H_S = \{\mathbf{x} \in \mathbb{R}^4 : x_3 = 0\}$ *strictly separates* $T_{\text{even}}^{\leq 4}(m)$ *from* $T_{\text{odd}}^{\leq 4}(m)$. *Moreover, we may choose the point of* $H_S \cap F_m^3$ *of lowest 4-coordinate to be* $\alpha_{m+1} = \operatorname{conv}\{\alpha_m, \beta_m\} \cap H_S$, *where* $(\alpha_{m+1})_4 = \tau_4 := (\tau_m)_4$.

Proof: Properties (M2) and (M3) are a matter of trivial affine transforms that can be chosen to leave the 4-coordinates invariant, thereby maintaining (M1), and property (M4) can be achieved via a translation and a shear $\sigma_{2,4}^a : x_2 \mapsto x_2 + ax_4$.

For (M5), choose $t \in F_m^2$ with $t_4 = q_4$ for some $q \in T_m^3$; such a point exists, since $\alpha_m \in F_m^2$, and $(\alpha_m)_4 < q_4' < \max\{s_4 : s \in F_m^2\}$ for all $q' \in T_m^3$ by (M1) and Remark 2.4. Translate t such that $t = (0, t_2, 0, 0)$ with $t_2 < 0$, and apply a shear transform $\sigma_{3,2}^b\colon x_3 \mapsto x_3 + bx_2$ to $\mathbb{R}^4$, where $b \in \mathbb{R}$ is chosen such that $\pi(\sigma_{3,2}^b(q)) = \pi(\sigma_{3,2}^b(t))$. This can be done because $\pi(t) - \pi(q) \in \mathbb{R}\pi(e_3)$. Then (M5) is fulfilled because $\dim \operatorname{aff} F_m^3 = 3$: supposing that $\dim \operatorname{aff}(\pi(F_m^2)) = 1$ would imply via $t \in F_m^2$ and $q \in F_m^3$ that $q \in \operatorname{aff} F_m^2$; however, this is absurd by the choice $q \in T_m^3$. Note that none of the maps we used affects (M2)–(M4).

For (M6), define $\tilde{b}$ to be the point of greatest 3-coordinate of $F_m^2 \cap \{\mathbf{x} \in \mathbb{R}^4 : x_4 = \tau_4\}$. In particular, $\tilde{b}_4 > \max_{z \in T^3} z_4$ by (M1), and $\tilde{b}$ lies either on the edge $\operatorname{conv}\{\alpha_m, \beta_m\}$ or on the edge $\operatorname{conv}\{\alpha_m, \omega_m\}$ of $F_m^2 \subset Q_m$ (cf. Figure 10).

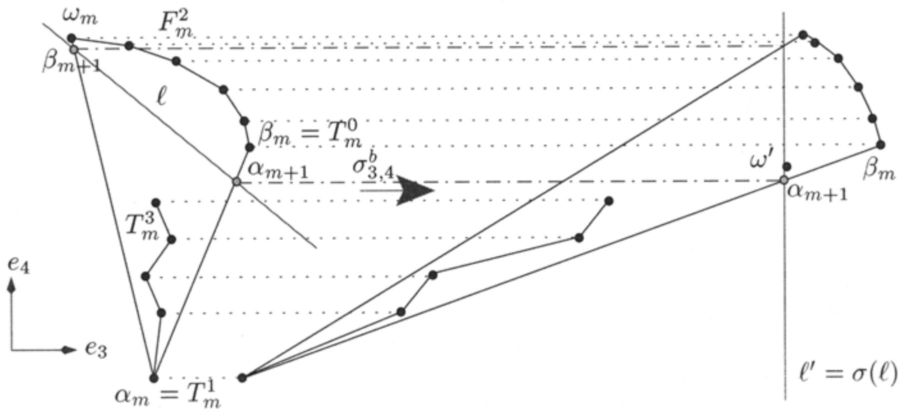

Figure 10. Positioning the polytope, step (M6). The map $\sigma_{3,4}^b$ shears the polytope until (the preimage under π of) a vertical line ℓ' separates the odd from the even tips. On the right, the approximate position of ω' is marked; cf. Lemma 5.7.

Possibly using the transform $x_3 \mapsto -x_3$, we can achieve $\tilde{b} \in \operatorname{conv}\{\alpha_m, \beta_m\}$, and $\tilde{b} = \alpha_{m+1}$ after a translation along the 3-axis. Now choose a non-horizontal line ℓ through α_{m+1} such that $\pi(\ell)$ separates $\pi(T_m^1 \cup T_m^3)$ from $\pi(F_m^2 \setminus T_m^1)$ (for example, perturb $\ell = \alpha_{m+1} + \mathbb{R}e_3$), translate Q_m again such that $\alpha_{m+1} = 0$, and apply a shear $\sigma_{3,4}^c\colon x_3 \mapsto x_3 + cx_4$ to $\mathbb{R}^4$ such that the line $\ell' := \sigma_{3,4}^c(\ell) = \{\mathbf{x} \in \mathbb{R}^4 : x_1 = x_3 = 0\} \cap \operatorname{aff} F_m^2$ is vertical, and $x_3 < 0 < y_3$ for all $x \in T_m^1 \cup T_m^3$ and $y \in F_m^2 \setminus T_m^1$ (cf. Figure 10). If the hyperplane $\pi^{-1}(\pi(\ell'))$ does not yet separate T_m^1 from T_m^4, apply another shear $\sigma_{3,1}^d\colon x_3 \mapsto x_3 + dx_1$ with $d > 0$

until it does (note that (M3) already holds), and then define $H_S := \pi^{-1}(\pi(\ell'))$. This hyperplane then separates the odd and even parts of π_m by construction, and $(\alpha_{m+1})_4 = \tau_4$ also by construction and because the shears $\sigma^c_{3,4}$ and $\sigma^d_{3,1}$ do not affect 4-coordinates. Neither do they affect conditions (M1)–(M5), so we define σ as the composition of all these maps. ∎

Remark 5.5: The conditions (M1)–(M6) are satisfied by the coordinates (5) for Q_0.

5.5. Induction step II: Finding the cutting plane. In this section, we will find a hyperplane H_{m+1} that gives rise to a polytope $Q'_{m+1} = Q_m \cap H^{\geq 0}_{m+1}$ of the same combinatorial type as $\widetilde{Q}_{m+1}$. Namely, assume that the conditions (M1)–(M6) hold, define H_{m+1} to be the hyperplane $\{\mathbf{x} \in \mathbb{R}^4 : \mathbf{n}^T\mathbf{x} = 0\}$ with $\mathbf{n} = (0, -\delta, 1, \varepsilon)^T$ for some small $\varepsilon \gg \delta > 0$, and assign the label $n+1 = m+d+2$ to H_{m+1}. Note that H_{m+1} converges to H_S as $\varepsilon, \delta \to 0$.

Remark 5.6: Up to now, we have put the facet F^3_m into the 3-plane $\{\mathbf{x} \in \mathbb{R}^4 : x_1 = 0\}$ and the tip T^4 into the half-space $\{\mathbf{x} \in \mathbb{R}^4 : x_1 > 0\}$. This allows us to move "almost all" of the vertices of π_m (namely, the portion inside T^4_m) "out of the way", via a shear $\sigma^a_{3,1}$ that only affects 3-coordinates. These "old" vertices will be dealt with in Lemma 5.8 below.

We still need to arrange for the first and last part of π_{m+1} to be traversed in the right order. We achieve this by adjusting the position of H_{m+1} via the parameters ε and δ in the definition of $\mathbf{n}$ (note that we chose $n_1 = 0$, because we are already done with T^4_m). If $\delta = 0$, then $\pi(H_{m+1})$ is a line whose slope is determined by ε. We choose $\varepsilon > 0$ to 'push out' the first part $T^1_{m+1} \cup T^3_{m+1}$ of the new path π_{m+1}. However, if we left $\delta = 0$ we would not correctly sweep the last portion $T^0_{m+1} \cup T^2_{m+1}$. Items (M8)–(M10) of Lemma 5.7 guarantee a correct sweep in Lemma 5.8 for sufficiently small $0 < \delta \ll \varepsilon$.

Lemma 5.7: *Assume conditions (M1)–(M6) and $(\alpha_{m+1})_3 = (\alpha_{m+1})_4 = 0$, and fix vertices $q \in T^{\leq 4}_{\text{odd}}(m)$ and $s \in T^{\leq 4}_{\text{even}}(m)$. Let $q' = \operatorname{conv}\{q, s\} \cap H_{m+1}$ be the intersection with H_{m+1} of the line through q and s (which is not necessarily an edge of Q_m). Then, if $a > 0$ is sufficiently large and $0 < \delta \ll \varepsilon$ are sufficiently small, the image $\sigma^a_{3,1}(Q_m)$ of Q_m under the shear $\sigma^a_{3,1}$ satisfies the following conditions (M7)–(M10); cf. also Figure 12 below.*

(M7) *$q'_3 > 0$ for $0 < \delta \ll \varepsilon$, and $q'_3 \searrow 0$ as $\delta, \varepsilon \searrow 0$. In other words, all points in $\sigma^a_{3,1}(Q_m) \cap H_{m+1}$ can be chosen to have positive 3-coordinate, but to lie arbitrarily close to H_S.*

(M8) *Set* $u := (v_1^{m+1})' = \mathrm{conv}\{\alpha_m, \tau_m\} \cap H_{m+1}$ *and suppose that* $q' \neq u$. *Then the image* $\pi(\mathrm{aff}\{u, q'\}) \subset \langle 3,4\rangle$ *of the line through* u *and* q' *under* π *comes arbitrarily close to being vertical as* $a \to \infty$ *and* $\varepsilon, \delta \to 0$.

(M9) *Set* $\alpha' := \alpha'_{m+1} = \mathrm{conv}\{\alpha_m, \beta_m\} \cap H_{m+1}$. *If it happens that* $q, \bar{q} \in T^3_m$ *and* $q_4 < \bar{q}_4$, *so that* $q', \bar{q}' \in T^3_{m+1}$ *and* $q'_4 < \bar{q}'_4$, *then the slope* $\sigma_{\alpha'\bar{q}'}$ *of the line* $\pi(\mathrm{aff}\{\alpha', \bar{q}'\})$ *is greater than the slope* $\sigma_{\alpha' q'}$ *of the line* $\pi(\mathrm{aff}\{\alpha', q'\})$ *(and both are negative).*

(M10) *Set* $\omega' := \omega'_{m+1} = \mathrm{conv}\{\beta_m, v_1^m\} \cap H_{m+1}$. *Then the slope* $\sigma_{\omega'\alpha'}$ *of the line* $\pi(\mathrm{aff}\{\omega', \alpha'\})$ *is less than the slope* $\sigma_{\omega' u}$ *of* $\pi(\mathrm{aff}\{\omega', u\})$.

Proof: We abbreviate $\sigma = \sigma^a_{3,1}$. For (M7), we have $\mathrm{conv}\{q, s\} \cap H_{m+1} \neq \emptyset$ since q and s are separated by H_{m+1} for small enough δ, ε. We calculate the intersection point $q' = \mathrm{conv}\{q, s\} \cap H_{m+1}$ by solving $\mathbf{n}^T\mathbf{q} + \mu\mathbf{n}^T(\mathbf{s} - \mathbf{q}) = 0$ for μ, obtaining

$$\mathbf{q}' = \mathbf{q} + \frac{\mathbf{n}^T\mathbf{q}}{\mathbf{n}^T(\mathbf{q} - \mathbf{s})}(\mathbf{s} - \mathbf{q}).$$

By (M2), the map σ leaves the points α', q, and ω' invariant, and maps s to $\sigma(\mathbf{s}) = \mathbf{s} + as_1\mathbf{e}_3$; as a consequence, $\mathbf{n}^T\sigma(\mathbf{s}) = \mathbf{n}^T\mathbf{s} + as_1$. Using $\mathbf{n}^T\mathbf{q} = -\delta q_2 + q_3 + \varepsilon q_4$, we obtain

$$\begin{aligned} \sigma(\mathbf{q}') &= \mathbf{q} + \frac{\mathbf{n}^T\mathbf{q}}{\mathbf{n}^T(\mathbf{q} - \mathbf{s}) - as_1}(\mathbf{s} - \mathbf{q} + as_1\mathbf{e}_3) \\ &\underset{a\to\infty}{\longrightarrow} \mathbf{q} + (0, 0, -\mathbf{n}^T\mathbf{q}, 0)^T = (0, q_2, \delta q_2 - \varepsilon q_4, q_4)^T. \end{aligned} \tag{6}$$

Because $q_4 < (\alpha_{m+1})_4 = 0$, we can choose $0 < \delta \ll \varepsilon$ so small that $\sigma(\mathbf{q}')_3 > 0$ (note that $q_2 \leq 0$ by (M4)). In particular, we obtain $\sigma(\mathbf{q}')_3 \searrow 0$ as $\varepsilon, \delta \searrow 0$.

Statement(M8) follows from (6) and the fact that

$$\lim_{a\to\infty} \frac{\sigma(\mathbf{q}')_4 - \sigma(\mathbf{u})_4}{\sigma(\mathbf{q}')_3 - \sigma(\mathbf{u})_3} = \frac{q_4 - u_4}{\delta(q_2 - u_2) - \varepsilon(q_4 - u_4)}.$$

For (M9), note that since α' is invariant under σ,

$$\sigma_{\alpha' q'} = \frac{\sigma(\mathbf{q}')_4 - \alpha'_4}{\sigma(\mathbf{q}')_3 - \alpha'_3} \underset{a\to\infty}{\longrightarrow} \frac{q_4 - \alpha'_4}{\delta q_2 - \alpha'_3 - \varepsilon q_4},$$

and similarly for $\bar{q}$; the statement now follows from $q_4 < \bar{q}_4$ and $0 < \delta \ll \varepsilon$.

To prove (M10), set $\alpha := \alpha_m$, $\beta := \beta_m$, $v := v_1^m$ and $\tau := \tau_m$. Then

$$\begin{aligned} u &= \mathrm{conv}\{\alpha, \tau\} \cap H_{m+1}, \\ \alpha' &= \mathrm{conv}\{\alpha, \beta\} \cap H_{m+1}, \quad \text{and} \\ \omega' &= \mathrm{conv}\{v, \beta\} \cap H_{m+1}. \end{aligned}$$

We need to verify that

$$\sigma_{\omega'\alpha'} := \frac{\alpha'_4 - \omega'_4}{\alpha'_3 - \omega'_3} < \frac{u_4 - \omega'_4}{u_3 - \omega'_3} =: \sigma_{\omega' u}.$$

From equation (6) and condition (M4), we deduce that

$$\lim_{a\to\infty} u = (0, 0, -\varepsilon\alpha_4, \alpha_4)^T.$$

For α' and ω' we obtain the following expressions:

$$\begin{aligned}
\alpha' &= \alpha + \frac{\mathbf{n}^T\alpha}{\mathbf{n}^T(\alpha-\beta)}(\beta-\alpha),\\
&= (0,0,\alpha_3,\alpha_4)^T + \frac{\alpha_3 + \varepsilon\alpha_4}{\delta\beta_2 + \alpha_3 - \beta_3 + \varepsilon(\alpha_4 - \beta_4)}\mathbf{w}_1,\\
\omega' &= \mathbf{v} + \frac{\mathbf{n}^T\mathbf{v}}{\mathbf{n}^T(\mathbf{v}-\beta)}(\beta - \mathbf{v})\\
&= (0, v_2, v_3, v_4)^T + \frac{-\delta v_2 + v_3 + \varepsilon v_4}{-\delta(v_2 - \beta_2) + v_3 - \beta_3 + \varepsilon(v_4 - \beta_4)}\mathbf{w}_2,
\end{aligned}$$

where we have set

$$\begin{aligned}
\mathbf{w}_1 &= (0, \beta_2, \beta_3 - \alpha_3, \beta_4 - \alpha_4)^T,\\
\mathbf{w}_2 &= (0, \beta_2 - v_2, \beta_3 - v_3, \beta_4 - v_4)^T.
\end{aligned}$$

For convenience, instead of (7) we will verify that $1/\sigma_{\omega'\alpha'} > 1/\sigma_{\omega' u}$. Indeed, expanding these expressions in terms of δ, ε, we obtain

$$\begin{aligned}
\frac{1}{\sigma_{\omega'\alpha'}} &= \frac{\beta_3 v_2 - \beta_2 v_3 + \overbrace{\alpha_3(\beta_2 - v_2)}^{t_1}}{v_3(\alpha_4 - \beta_4) + \beta_3(v_4 - \alpha_4) + \underbrace{\alpha_3(\beta_4 - v_4)}_{t_2}}\delta - \varepsilon + p_1(\delta,\varepsilon),\\
\frac{1}{\sigma_{\omega' u}} &= \frac{\beta_3 v_2 - \beta_2 v_3}{v_3(\alpha_4 - \beta_4) + \beta_3(v_4 - \alpha_4)}\delta - \varepsilon + p_2(\delta, \varepsilon),
\end{aligned}$$

where p_1 and p_2 are power series in δ, ε with min-degree at least 2. Notice that up to terms of degree at least 2 in δ, ε, the two formulas are equal except for the expressions t_1 resp. t_2 in the numerator resp. denominator of $1/\sigma_{\omega'\alpha'}$. Therefore, we can write the difference between the inverses of the slopes as

$$\frac{1}{\sigma_{\omega'\alpha'}} - \frac{1}{\sigma_{\omega' u}} = \Big(\frac{A + t_1}{B + t_2} - \frac{A}{B}\Big)\delta + p_3(\delta, \varepsilon).$$

Since $\alpha_3 < (\alpha_{m+1})_3 < 0$ by assumption and $\beta_2 < v_2$ by (M4), we obtain $t_1 > 0$; and the inductive assumption (M1) implies that $\beta_4 > v_4$ and therefore $t_2 < 0$. The claim follows. ∎

5.6. Induction step III: The projective transformation. As a last step, we construct a 1-parameter family $\mathcal{H} = \{H_t : t \in \mathbb{P}^1(\mathbb{R})\}$ of hyperplanes that contains a 2-plane R as their common "axis", as in Section 5.1. Let

$$O = \pi(b + \varepsilon_1(\omega - \alpha) - \varepsilon_3 e_3)$$

for some small $\varepsilon_1, \varepsilon_3 > 0$, so that O lies outside but very close to the edge $\operatorname{conv}\{\alpha, \omega\}$ of $\pi(F^2_{m+1})$, and define the 2-plane $R \subset \mathbb{R}^4$ to be $R = \pi^{-1}(O)$.

Lemma 5.8: *Let $\mathcal{H}$ be the pencil of hyperplanes in $\mathbb{R}^4$ sharing the 2-plane R, and such that $\pi(H_\infty)$ is the line through O parallel to $\operatorname{conv}\{\alpha, \omega\}$, and the slope of $\pi(H_r)$ is smaller than the slope of $\pi(H_s)$ exactly if $r < s$. Then $\mathcal{H}$ fulfills (S2), i.e., it sorts the vertices of Q_{m+1} in the order given by π_{m+1}.*

Proof: We examine the pieces of π_{m+1} in order; cf. Figure 12.

- ▷ *$T^1_{m+1} = \{\alpha\}$ is the start of π_{m+1}:* This follows for small enough ε_3 by (M10).
- ▷ *T^3_{m+1} is traversed next, in the right order, and before T^4_{m+1}:* The first two statements follow from (M7), (M8) and (M9), and the last one because $z_3 \to \infty$ as $a \to \infty$ for any $z \in T^4_m$, while the 3-coordinates of T^3_{m+1} remain bounded by (M7).
- ▷ *The correct order in $T^4_m \subset T^4_{m+1}$.* By Proposition 5.2(b), each of the edges $e_i = \operatorname{conv}\{v^m_i, v^m_{i+1}\}$, $0 \le i \le m$, of $T^1_m \cup T^3_m$ is incident to an $(m+1)$-gonal 2-face G_i (see Figure 9), and the edges E_i of G_i not incident to e_i form a monotone subpath of π_{m+1}. This implies that for each $e_i \in T^3_m$, the slopes of the projection of each E_i to $\langle 3,4\rangle$ are strictly positive (and, by convexity, monotonically decreasing; see Figure 11).
  Therefore, $\pi(\bigcup_{i=0}^m E_i)$ is a strictly increasing chain of edges. This fact remains true after applying the linear map $\sigma = \sigma^a_{3,1}$ by invariance of the e_i's and all 4-coordinates under σ, and the convexity of the projections of 2-faces. The correct order up to τ in $T^4_m \subset T^4_{m+1}$ follows from condition (M6): $\alpha_4 \ge s_4$ for all $s \in \bigcup_{i=0}^m \operatorname{vert} G_i \setminus \operatorname{vert} e_i$. Similarly, the 4-gonal 2-faces incident to T^2_m of Proposition 5.2(c) enforce the right order between τ and T^0_m.
- ▷ *T^2_{m+1} is traversed after T^4_{m+1}:* Since β, the first vertex of π_{m+1} to come after T^4_{m+1}, lies on $\operatorname{conv}\{\alpha_m, \omega_m\}$, this can be achieved by choosing ε and ε_1 suitably small.
- ▷ *Correct order in T^2_{m+1} and T^0_{m+1}.* This follows because the convex polygon $\pi(F^2_{m+1})$ is star-shaped with respect to any point on its boundary, and the

choice of O close to an edge of $\pi(F^2_{m+1})$.

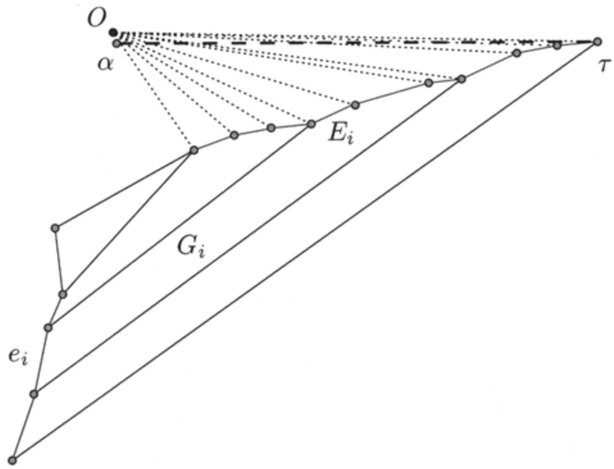

Figure 11. Convexity of the $(m+1)$-gonal faces G_i enforces the correct order in $T^4_m \subset T^4_{m+1}$.

This concludes the proof of Lemma 5.8. ∎

Finally, we apply the projective transform ψ: $\mathbb{R}^4 \to \mathbb{R}^4$, $\mathbf{x} \mapsto \mathbf{x}/(\mathbf{ax} - a_0)$ that sends the 3-plane $H_\infty = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{ax} = a_0\}$ to infinity, and set $Q_{m+1} := \psi(Q'_{m+1})$. Lemma 5.8 then implies the inductive condition (M1), namely that Q_{m+1} admits a monotone Hamilton path π_{m+1}. The proof of Theorem 2.5, and so of the Main Theorem, is concluded. ∎

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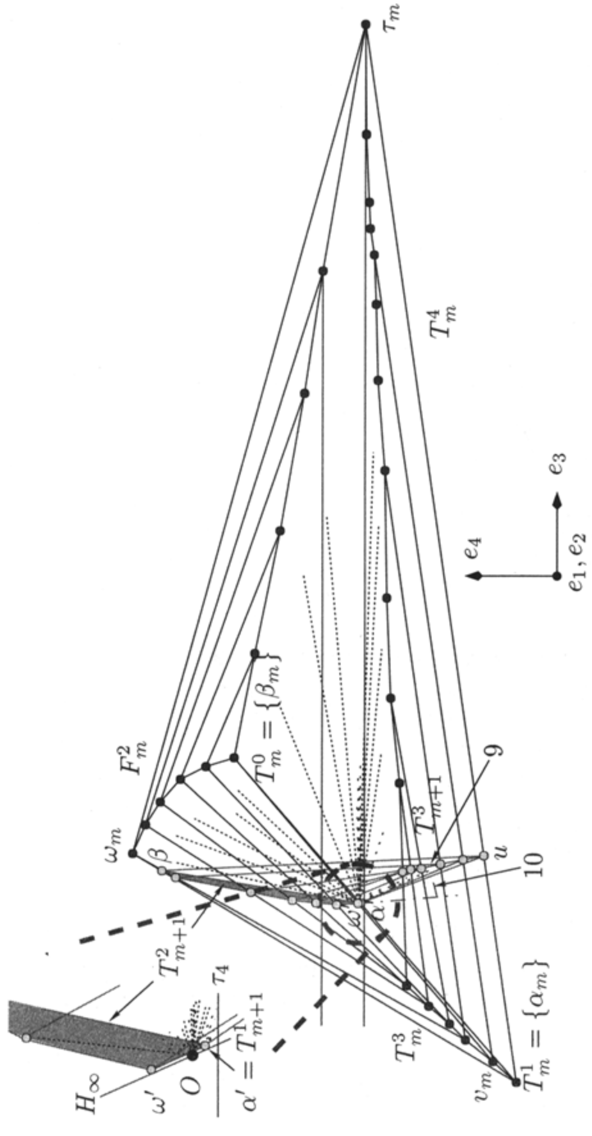

Figure 12. *The inductive step:* We show the projection of the polytope Q_4 to the $\langle 3,4\rangle$-plane, and the vertices obtained by intersecting Q_4 with H_5. The arrows next to the labels 9 and 10 point to the lines about whose slope the corresponding condition in Lemma 5.7 makes an assertion. The line through O is the projection of the 3-plane H_∞. A sweep around O encounters all vertices of $Q_m \cap H_{m+1}$ in the correct order π_m prescribed by $\widetilde{\pi}_{m+1}$.

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